

A NEW EXPANDER AND IMPROVED BOUNDS FOR $A(A + A)$

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ABSTRACT. The main result in this paper concerns a new five-variable expander. It is proven that for any finite set of real numbers A ,

$$|\{(a_1 + a_2 + a_3 + a_4)^2 + \log a_5 : a_1, a_2, a_3, a_4, a_5 \in A\}| \gg \frac{|A|^2}{\log |A|}.$$

This bound is optimal, up to logarithmic factors. The paper also gives new lower bounds for $|A(A - A)|$ and $|A(A + A)|$, improving on results from [8]. The new bounds are

$$|A(A - A)| \gtrsim |A|^{3/2 + \frac{1}{34}}$$

and

$$|A(A + A)| \gtrsim |A|^{3/2 + \frac{5}{242}}.$$

1. INTRODUCTION

Throughout this paper, the standard notation \ll, \gg is applied to positive quantities in the usual way. Saying $X \gg Y$ means that $X \geq cY$, for some absolute constant $c > 0$. All logarithms in the paper are base 2. We use the symbols \lesssim, \gtrsim to suppress both constant and logarithmic factors. To be precise, we write $X \gtrsim Y$ if there is some absolute constant $c > 0$ such that $X \gg Y/(\log X)^c$.

This paper is concerned with a particular variation of the sum-product problem. A fundamental idea in sum-product theory is that a finite set A in a field \mathbb{F} cannot be highly structured in both an additive and multiplicative sense. This is a guiding principle behind the Erdős-Szemerédi sum-product conjecture, which states that for any finite set $A \subset \mathbb{Z}$ and all $\epsilon > 0$,

$$\max\{|A + A|, |AA|\} \geq c_\epsilon |A|^{2-\epsilon},$$

where $A + A := \{a + b : a, b \in A\}$ is the *sum set*, and the *product set* AA is defined analogously. The conjecture remains wide open. See the recent work of Konyagin and Shkredov [4] for the most up to date bounds, and the references within for more background on this famous problem. See also chapter 8 of [14] for a more detailed, although now slightly outdated, introduction to the sum-product problem.

By the same principle, it is typically expected that a set which is defined by a combination of additive and multiplicative operations on elements of an input set A will always be large compared to A . For example, it follows from an ingenious geometric argument of Ungar [15]

that for any finite set $A \subset \mathbb{R}$,

$$(1.1) \quad \left| \frac{A - A}{A - A} \right| \geq |A|^2 - 2,$$

where

$$\frac{A - A}{A - A} := \left\{ \frac{a - b}{c - d} : a, b, c, d \in A, c \neq d \right\}.$$

We say that a function¹ $f : D \rightarrow \mathbb{R}$ is a d -variable expander if $D \subset \mathbb{R}^d$ and it is true that there is some $\epsilon > 0$ such that for any finite $A \subset \mathbb{R}$

$$|\{f(a_1, \dots, a_d) : a_i \in A\}| \gg |A|^{1+\epsilon}.$$

So, inequality (1.1) shows that the function $f(x_1, x_2, x_3, x_4) = \frac{x_1 - x_2}{x_3 - x_4}$ is a 4-variable expander.

Results on expanders which are tight up to constant and logarithmic factors are relatively rare. For 3 variables, the only such result is due to Jones [3], who proved that

$$(1.2) \quad \left| \left\{ \frac{a - c}{a - b} : a, b, c \in A \right\} \right| \gg \frac{|A|^2}{\log |A|}.$$

A slightly different proof of this inequality can also be found in [10]. For 4 variables, as well as the aforementioned result (1.1), it is known that

$$(1.3) \quad |\{(a - b)^2 + (c - d)^2 : a, b, c, d \in A\}| \gg \frac{|A|^2}{\log |A|}$$

and

$$(1.4) \quad |(A + A)(A + A)| \gg \frac{|A|^2}{\log |A|}.$$

The results are due to Guth and Katz [2] and Roche-Newton and Rudnev [11] respectively. A considerably more simple proof of (1.4) was given in [10]. It was also established by Balog and Roche-Newton [1] that

$$(1.5) \quad \left| \frac{A + A}{A + A} \right| \geq 2|A|^2 - 1.$$

The following 5-variable expander result was proven by Murphy, Roche-Newton and Shkredov [8]:

$$(1.6) \quad |A(A + A + A + A)| \gg \frac{|A|^2}{\log |A|}.$$

All of these results are optimal up to constant and logarithmic factors, as can be seen by taking A to be an arithmetic progression, and this gives a complete list of the known optimal expander results for 5 or less variables. In this paper, we add to the list by proving the following theorem:

¹We usually take $D = \mathbb{R}^d$, but we use a general $D \subset \mathbb{R}^d$ in this definition to avoid the possibility of dividing by zero.

Theorem 1.1. *For any finite set of real numbers A ,*

$$|\{(a_1 + a_2 + a_3 + a_4)^2 + \log a_5 : a_i \in A\}| \gg \frac{|A|^2}{\log |A|}.$$

This gives an optimal bound, up to log factors, for the admittedly curious expander function

$$f(a_1, a_2, a_3, a_4, a_5) = (a_1 + a_2 + a_3 + a_4)^2 + \log a_5.$$

This paper also considers the more natural expander $f(a, b, c) = a(b - c)$. It is easy to use the Szemerédi-Trotter Theorem to prove that $|A(A - A)| \gg |A|^{3/2}$. See [14, Exercise 8.3.3] for a similar result. In [8], this was improved to $|A(A - A)| \gg |A|^{\frac{3}{2} + \frac{1}{112}}$. Here, we improve this further:

Theorem 1.2. *For any finite set A of real numbers*

$$|A(A - A)| \gtrsim |A|^{\frac{3}{2} + \frac{1}{34}}.$$

Similarly, we prove the following result for the expander $f(a, b, c) = a(b + c)$.

Theorem 1.3. *For any finite set A of real numbers*

$$|A(A + A)| \gtrsim |A|^{\frac{3}{2} + \frac{5}{242}}.$$

This improves the bound $|A(A + A)| \gg |A|^{\frac{3}{2} + \frac{1}{178}}$ from [8]. The proofs of these two theorems use ideas from a recent paper of Konyagin and Shkredov [4] to streamline the original argument by avoiding using the Balog-Szemerédi-Gowers Theorem.

2. NOTATION AND PRELIMINARY RESULTS

Given finite sets $A, B \subset \mathbb{R}$, the *additive energy of A and B* is the number of solutions to the equation

$$a_1 - b_1 = a_2 - b_2$$

such that $a_1, a_2 \in A$ and $b_1, b_2 \in B$. The additive energy is denoted $E^+(A, B)$. Let

$$r_{A-B}(x) := |\{(a, b) \in A \times B : a - b = x\}|.$$

Note that $r_{A-B}(x) = |A \cap (B + x)|$. The notation of the representation function r will be used with flexibility throughout this paper, with the information about the kind of representations it counts being contained in a subscript. For example,

$$r_{(A-A)^2 + (A-A)^2}(x) = |\{(a_1, a_2, a_3, a_4) \in A^4 : (a_1 - a_2)^2 + (a_3 - a_4)^2 = x\}|.$$

Note that

$$E^+(A, B) = \sum_{x \in A-B} r_{A-B}^2(x).$$

The shorthand $E^+(A, A) = E^+(A)$ is used. The notion of energy can be extended to an arbitrary power k . We define $E_k^+(A)$ by the formula

$$E_k^+(A) = \sum_{x \in A-A} r_{A-A}^k(x).$$

Similarly, the *multiplicative energy* of A and B , denoted $E^*(A, B)$, is the number of solutions to the equation

$$\frac{a_1}{b_1} = \frac{a_2}{b_2},$$

such that $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

The notions of additive and multiplicative energy have been central in the literature on sum-product estimates. For example, the key ingredient in the beautiful work of Solymosi [12], which until recently held the record for the best known sum-product estimate, is the following bound:

Theorem 2.1. *For any finite set $A \subset \mathbb{R}$,*

$$E^*(A) \ll |A + A|^2 \log |A|.$$

We will also call upon the following result on the relationship between different types of energy:

Lemma 2.1 ([6], Lemma 2.4 and Lemma 2.5). *For any finite sets $A, B \subset \mathbb{R}$,*

$$|A|^2 (E_{1.5}^+(A))^2 \leq (E_3^+(A))^{2/3} (E_3^+(B))^{1/3} E(A, A - B).$$

In [8], the following lemma played an important role:

Lemma 2.2. *For any finite sets $A, B, C \subset \mathbb{R}$,*

$$E^*(A) |A(B + C)|^2 \gg \frac{|A|^4 |B| |C|}{\log |A|}.$$

The proof in [8] uses only the Szemerédi-Trotter Theorem, but it can also be proved in a more superficially straightforward way by using ideas from the work of Guth and Katz [2] on the Erdős distinct distance problem. In particular, Lemma 2.2 follows very easily if we assume the following result from [11]:

Theorem 2.2. *For any finite set $A \subset \mathbb{R}$, the number of solutions to the equation*

$$(a_1 - a_2)(a_3 - a_4) = (a_5 - a_6)(a_7 - a_8)$$

such that $a_1, \dots, a_8 \in A$ is $O(|A|^6 \log |A|)$.

The same result holds with the minus signs changed to plus signs. The proof of Theorem 2.2 in [11] closely follows the work of Guth and Katz, and is an analogue of the following result from [2]:

Theorem 2.3. *For any finite set $A \subset \mathbb{R}$, the number of solutions to the equation*

$$(a_1 - a_2)^2 + (a_3 - a_4)^2 = (a_5 - a_6)^2 + (a_7 - a_8)^2$$

such that $a_1, \dots, a_8 \in A$ is $O(|A|^6 \log |A|)$.

Once again, the same bound holds for the equation

$$(a_1 + a_2)^2 + (a_3 + a_4)^2 = (a_5 + a_6)^2 + (a_7 + a_8)^2.$$

In fact, Theorems 2.3 and 2.2 are special cases of more general geometric results which were proved respectively in [2] and [11], but here they are stated only in the forms in which they will be used in this paper.

Theorem 2.3 can be used to prove the following variation of Lemma 2.2:

Lemma 2.3. *For any finite sets $A, B \in \mathbb{R}$,*

$$E^+(A)|\{a + (b_1 + b_2)^2 : a \in A, b_1, b_2 \in B\}|^2 \gg \frac{|A|^4|B|^2}{\log |B|}.$$

Proof. The proof proceeds by the familiar method of double counting the number of solutions to the equation

$$(2.1) \quad a_1 + (b_1 + b_2)^2 = a_2 + (b_3 + b_4)^2$$

such that $a_i \in A$ and $b_i \in B$. Let S denote the number of solutions to (2.1) and write

$$A + (B + B)^2 := \{a + (b_1 + b_2)^2 : a \in A, b_1, b_2 \in B\}.$$

By the Cauchy-Schwarz inequality

$$S \geq \frac{|A|^2|B|^4}{|A + (B + B)^2|}.$$

On the other hand, also by the Cauchy-Schwarz inequality

$$\begin{aligned} S^2 &= \left(\sum_x r_{A-A}(x) r_{(B+B)^2-(B+B)^2}(x) \right)^2 \\ &\leq \left(\sum_x r_{A-A}^2(x) \right) \left(\sum_x r_{(B+B)^2-(B+B)^2}^2(x) \right) \\ &= E^+(A) \left(\sum_x r_{(B+B)^2-(B+B)^2}^2(x) \right) \end{aligned}$$

Theorem 2.3 tells us that $\left(\sum_x r_{(B+B)^2-(B+B)^2}^2(x) \right) = O(|B|^6 \log |B|)$. Therefore

$$\begin{aligned} |A|^4|B|^8 &\leq |A + (B + B)^2|^2 S^2 \\ &\ll |A + (B + B)^2|^2 E^+(A) |B|^6 \log |B|. \end{aligned}$$

After rearranging this inequality, we obtain the desired result. \square

Unfortunately, we are not aware of a proof of Lemma 2.3 which does not use the deep results from [2].

3. FIVE VARIABLE EXPANDER

It is now straightforward to use the results from the previous section to prove the result on the new five variable expander.

Theorem 3.1. *For any finite set $A \subset \mathbb{R}$*

$$|\{(a_1 + a_2 + a_3 + a_4)^2 + \log a_5 : a_i \in A\}| \gg \frac{|A|^2}{\log |A|}.$$

Proof. Apply Lemma 2.3 with $A = \log A$ and $B = A + A$. We have

$$E^+(\log(A))|\{(a_1 + a_2 + a_3 + a_4)^2 + \log a_5 : a_i \in A\}|^2 \gg \frac{|A|^4|A + A|^2}{\log |A|}.$$

Note that $\log a_1 + \log a_2 = \log a_3 + \log a_4$ if and only if $a_1 a_2 = a_3 a_4$, and so $E^+(\log(A)) = E^*(A)$. We can apply Theorem 2.1 to deduce that

$$E^+(\log A) \ll |A + A|^2 \log |A|.$$

It then follows that

$$|\{(a_1 + a_2 + a_3 + a_4)^2 + \log a_5 : a_i \in A\}|^2 \gg \frac{|A|^4}{\log^2 |A|},$$

which completes the proof. □

4. THREE VARIABLE EXPANDERS

In a recent paper of Konyagin and Shkredov [4], a new characteristic for a finite set of real numbers A was considered. Define $d_*(A)$ by the formula

$$d_*(A) = \min_{t>0} \min_{\emptyset \neq Q, R \subset \mathbb{R} \setminus \{0\}} \frac{|Q|^2 |R|^2}{|A| t^3},$$

where the second minimum is taken over all Q and R such that $\max\{|Q|, |R|\} \geq |A|$ and such that for every $a \in A$, the bound $|Q \cap aR^{-1}| \geq t$ holds. Konyagin and Shkredov proved the following lemma:

Lemma 4.1 (Lemma 13, [4]). *For any $A, B \subset \mathbb{R}$ and any $\tau \geq 1$,*

$$|\{x : r_{A-B}(x) \geq \tau\}| \ll \frac{|A||B|^2}{\tau^3} d_*(A).$$

The proof uses the Szemerédi-Trotter Theorem. Lemma 4.1 generalises an earlier result in which the bound

$$(4.1) \quad |\{x : r_{A-B}(x) \geq \tau\}| \ll \frac{|A||B|^2}{\tau^3} d(A)$$

was established, where $d(A) = \min_{C \neq \emptyset} \frac{|AC|^2}{|A||C|}$. See [9, Lemma 7] for a proof. As pointed out in [4], $d_*(A) \leq d(A)$, since for any non empty C we can take $t = |C|$, $Q = AC$ and $R = C^{-1}$ in the definition of $d_*(A)$.

Using the language of [4] and [13], we could rephrase Lemma 4.1 by saying that A is a Szemerédi-Trotter set with $O(d_*(A))$.

We can use Lemma 4.1 to prove the following lemma. No originality is claimed here - we are essentially copying the arguments from [9] and predecessors with the stronger Lemma 4.1 in place of the bound (4.1) - but we include the proof for completeness.

Lemma 4.2. *For any finite set $A \subset \mathbb{R}$,*

$$|A - A| \gg \frac{|A|^{8/5}}{d_*^{3/5}(A) \log^{2/5} |A|}.$$

Proof. First, we will prove two energy bounds. Note that, by Lemma 4.1,

$$(4.2) \quad \begin{aligned} E_3^+(A) &= \sum_x r_{A-A}^3(x) \\ &= \sum_{j \geq 1} \sum_{x: 2^{j-1} \leq r_{A-A}(x) < 2^j} r_{A-A}^3(x) \\ &\ll |A|^3 d_*(A) \log |A|. \end{aligned}$$

Similarly, for any $F \subset \mathbb{R}$,

$$\begin{aligned} E^+(A, F) &= \sum_x r_{A-F}^2(x) \\ &= \sum_{x: r_{A-F} < \Delta} r_{A-F}^2(x) + \sum_{j \geq 1} \sum_{x: \Delta 2^{j-1} \leq r_{A-F}(x) < \Delta 2^j} r_{A-F}^2(x) \\ &\ll \Delta |A| |F| + \frac{|A| |F|^2 d_*(A)}{\Delta}. \end{aligned}$$

We choose $\Delta = (|F| d_*(A))^{1/2}$, and thus conclude that

$$(4.3) \quad E(A, F) \ll |A| |F|^{3/2} d_*(A)^{1/2}.$$

Now, by Hölder's inequality,

$$\begin{aligned} |A|^6 &= \left(\sum_{x \in A-A} r_{A-A}(x) \right)^3 \\ &\leq \left(\sum_x r_{A-A}^{3/2} \right)^2 |A-A| \\ &= (E_{1.5}^+(A))^2 |A-A|. \end{aligned}$$

Finally, applying Lemma 2.1 as well as inequalities (4.2) and (4.3), we have

$$\begin{aligned} |A|^8 &\leq E_3^+(A) E^+(A, A-A) |A-A| \\ &\ll |A|^4 |A-A|^{5/2} d_*^{3/2}(A) \log |A|, \end{aligned}$$

and it follows that

$$|A-A| \gg \frac{|A|^{8/5}}{d_*^{3/5}(A) \log^{2/5} |A|}.$$

□

The following similar result for sum sets follows from a combination of the work in [13] and [4]:

Lemma 4.3. *For any finite set $A \subset \mathbb{R}$,*

$$|A+A| \gtrsim \frac{|A|^{58/37}}{d_*^{21/37}(A)}.$$

To be more precise, it was proven in [13] that if A is a Szemerédi-Trotter set with D , then $|A+A| \gtrsim \frac{|A|^{58/37}}{D^{21/37}}$, and it was subsequently established in [4] that any set A is a Szemerédi-Trotter set with $O(d_*(A))$.

The methods used in [13] are rather different from those used in this paper, and appear to be far from trivial. However, one can obtain a quantitatively weaker bound

$$(4.4) \quad |A+A| \gg \frac{|A|^{14/9}}{d_*^{5/9}(A) \log^{2/9} |A|}$$

with a proof very similar to that of Lemma 4.2 above. To see how this works, one can repeat the arguments from the proof of Theorem 1.2 in [7], but using Lemma 4.1 in place of Lemma 3.2 from [7]. This is worth noting, since the proofs of the main results in [5] and [4], that is the bound

$$\max\{|A+A|, |AA|\} \gg |A|^{4/3+c},$$

for some $c > 0$, both include applications of Lemma 4.3. One can also obtain this sum-product estimate, albeit with a smaller value of c , by using the bound (4.4) instead of Lemma 4.3.

The Balog-Szemerédi-Gowers Theorem tells us that if a set A has large multiplicative energy, then there is a large subset $A' \subset A$ such that $|A'/A'|$ is small. We can then use sum-product theory to deduce that $A' + A'$, and thus also $A + A$, is large. The following result arrives at the same conclusion, but avoids applying the Balog-Szemerédi-Gowers Theorem, and therefore gives quantitatively better estimates for the problem at hand. The proof uses ideas from [4].

Lemma 4.4. *Let $A \subset \mathbb{R}$ and suppose that $E^*(A) \geq \frac{|A|^3}{K}$. Then*

$$|A - A| \gtrsim \frac{|A|^{8/5}}{K^{6/5}}$$

and

$$|A + A| \gtrsim \frac{|A|^{58/37}}{K^{42/37}}.$$

Proof. The idea here is to use the hypothesis that the energy is large in order to find a large subset $A' \subset A$ such that $d_*(A)$ is small, and to then apply Lemmas 4.2 and 4.3 to complete the proof.

We can write

$$E^*(A) = \sum_x |A \cap xA|^2.$$

Note that

$$\sum_{x: |A \cap xA| \leq \frac{E^*(A)}{2|A|^2}} |A \cap xA|^2 \leq \frac{E^*(A)}{2|A|^2} \sum_x |A \cap xA| \leq \frac{E^*(A)}{2}$$

and so

$$\sum_{x: |A \cap xA| \geq \frac{E^*(A)}{2|A|^2}} |A \cap xA|^2 \geq \frac{E^*(A)}{2}.$$

Therefore, by a dyadic pigeonholing argument, there exists $\frac{E^*(A)}{2|A|^2} \leq \tau \leq |A|$ such that

$$\sum_{x: \tau \leq |A \cap xA| < 2\tau} |A \cap xA|^2 \gtrsim E^*(A).$$

We label $S_\tau := \{x : \tau \leq |A \cap xA| < 2\tau\}$, and thus we have

$$(4.5) \quad |S_\tau| \tau^2 \gtrsim E^*(A).$$

Observe that

$$(4.6) \quad \sum_{a \in A} |A \cap aS_\tau| = \sum_{x \in S_\tau} |A \cap xA| \geq |S_\tau| \tau.$$

Now apply another dyadic pigeonholing argument to obtain a subset $A' \subset A$ such that for all $a \in A'$

$$(4.7) \quad t \leq |A \cap aS_\tau| < 2t,$$

for some real number $0 < t \leq |A|$, and such that

$$\sum_{a \in A'} |A \cap aS_\tau| \gtrsim |S_\tau|\tau.$$

Therefore

$$(4.8) \quad |A'|t \gtrsim |S_\tau|\tau.$$

Note, since $t \leq |A|$, that

$$(4.9) \quad |A'| \gtrsim \frac{|S_\tau|\tau}{|A|}.$$

For every $a \in A'$, we have $|A \cap aS_\tau| \geq t$. Therefore, we can take

$$t = t, Q = A, R = S_\tau^{-1}$$

in the definition of $d_*(A')$. We then have

$$(4.10) \quad d_*(A') \leq \frac{|A|^2|S_\tau|^2}{|A'|t^3}.$$

Apply Lemma 4.2 to get

$$\begin{aligned} |A - A| &\geq |A' - A'| \\ &\gtrsim \frac{|A'|^{8/5}}{d_*^{3/5}(A')} \\ &\gg |A'|^{8/5} \left(\frac{|A'|t^3}{|A|^2|S_\tau|^2} \right)^{3/5} \\ &\gtrsim \frac{(|S_\tau|\tau)^{9/5}|A'|^{2/5}}{|A|^{6/5}|S_\tau|^{6/5}} \\ &\gtrsim \frac{(|S_\tau|\tau)^{9/5} \left(\frac{|S_\tau|\tau}{|A|} \right)^{2/5}}{|A|^{6/5}|S_\tau|^{6/5}} \\ &= \frac{|S_\tau|\tau^{11/5}}{|A|^{8/5}} \\ &\gtrsim \frac{E^*(A) \left(\frac{E^*(A)}{|A|^2} \right)^{1/5}}{|A|^{8/5}} \\ &\geq \frac{\left(\frac{|A|^3}{K} \right)^{6/5}}{|A|^2} = \frac{|A|^{8/5}}{K^{6/5}}. \end{aligned}$$

Similarly, an application of Lemma 4.3 gives

$$\begin{aligned}
|A + A| &\geq |A' + A'| \\
&\gtrsim \frac{|A'|^{58/37}}{d_*^{21/37}(A')} \\
&\gg |A'|^{58/37} \left(\frac{|A'|t^3}{|A|^2|S_\tau|^2} \right)^{21/37} \\
&\gtrsim \frac{(|S_\tau|\tau)^{63/37}|A'|^{16/37}}{|A|^{42/37}|S_\tau|^{42/37}} \\
&\gtrsim \frac{(|S_\tau|\tau)^{63/37} \left(\frac{|S_\tau|\tau}{|A|} \right)^{16/37}}{|A|^{42/37}|S_\tau|^{42/37}} \\
&= \frac{|S_\tau|\tau^{79/37}}{|A|^{58/37}} \\
&\gtrsim \frac{E^*(A) \left(\frac{E^*(A)}{|A|^2} \right)^{5/37}}{|A|^{58/37}} \\
&\geq \frac{\left(\frac{|A|^3}{K} \right)^{42/37}}{|A|^{68/37}} = \frac{|A|^{58/37}}{K^{42/37}}.
\end{aligned}$$

□

We are now ready to prove the new lower bounds for $|A(A - A)|$ and $|A(A + A)|$.

Theorem 4.1. *For any finite set $A \subset \mathbb{R}$,*

$$|A(A - A)| \gtrsim |A|^{\frac{3}{2} + \frac{1}{34}}.$$

Proof. Write $E^*(A) = \frac{|A|^3}{K}$. The proof considers two cases:

Case 1 - Suppose that $K \geq |A|^{1/17}$. Then apply Lemma 2.2, which tells us that

$$\frac{|A|^6}{\log |A|} \leq E^*(A)|A(A - A)|^2 = \frac{|A|^3|A(A - A)|^2}{K}$$

and therefore

$$|A(A - A)| \gtrsim |A|^{3/2} K^{1/2} \geq |A|^{\frac{3}{2} + \frac{1}{34}}.$$

Case 2 - Suppose that $K \leq |A|^{1/17}$. Then, by Lemma 4.4,

$$|A(A - A)| \geq |A - A| \gtrsim \frac{|A|^{8/5}}{K^{6/5}} \geq |A|^{\frac{8}{5} - \frac{6}{85}} = |A|^{\frac{3}{2} + \frac{1}{34}}.$$

□

Theorem 4.2. *For any finite set $A \subset \mathbb{R}$,*

$$|A(A + A)| \gtrsim |A|^{\frac{3}{2} + \frac{5}{242}}.$$

Proof. Write $E^*(A) = \frac{|A|^3}{K}$. The proof considers two cases:

Case 1 - Suppose that $K \geq |A|^{5/121}$. Then Lemma 2.2 tells us that

$$|A(A + A)| \gtrsim |A|^{3/2} K^{1/2} \geq |A|^{\frac{3}{2} + \frac{5}{242}}.$$

Case 2 - Suppose that $K \leq |A|^{5/121}$. Then, by Lemma 4.4,

$$|A(A + A)| \geq |A + A| \gtrsim \frac{|A|^{58/37}}{K^{42/37}} \geq |A|^{\frac{58}{37} - \frac{5 \cdot 42}{121 \cdot 37}} = |A|^{\frac{3}{2} + \frac{5}{242}}.$$

□

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